MATHEMATICS

B. Sc, Part I (Paper II)

Topic - Integration

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33. Integration of $\int \frac{1}{L\sqrt{M}} dx$.

Note The following rules are also applicable in the case $\int \frac{\phi(x)}{L\sqrt{M}} dx$

(A) If L and M are both linear, we put $\sqrt{M}=z$.

The following example will illustrate the above method.

Ex. *Integrate* $\int \frac{dx}{(x-3)\sqrt{x+1}}$

we put $\sqrt{x+1}=z$ so that $x+1=z^2$ $\therefore$ $dx=2zdz$

and $x=z^2-1$, $x-3=z^2-1-3=z^2-4$

$$\therefore \quad I=\int \frac{dx}{(x-3)\sqrt{x+1}}=\int \frac{2zdz}{(z^2-4)z}=2\int \frac{dz}{z^2-4}$$

$$=2.\frac{1}{2.2}\log\frac{z-2}{z+2}=\tfrac{1}{2}\log\frac{z-2}{z+2}$$

$$=\tfrac{1}{2}\log\frac{\sqrt{x+1}-2}{\sqrt{x+1}+2}.$$

(B) If L is quadratic and M is linear, we put $\sqrt{M}=z$

The following example will illustrate the method.

Ex. *Integrate* $\int \frac{x+2}{2(x^2+9x+9)\sqrt{x+1}} dx$

Let. $\sqrt{x+1}=z$... or, $x+1=z^2$

$\therefore$ $dx=2zdz$, $x=z^2-1$ or, $x+2=z^2+1$.

$$\text{Thus,}\quad I=\int \frac{(z^2+1)\,2zdz}{2\{(z^2-1)^2+9(z^2-1)+9\}z}$$

$$=\int \frac{(z^2+1)\,dz}{(z^4-2z^2+1+9z^2-9+9)}$$

$$=\int\frac{(z^2+1)\,dz}{z^4+7z^2+1}=\int\frac{1+\frac{1}{z^2}}{z^2+7+\frac{1}{z^2}}\,dz$$

$$=\int\frac{\left(1+\frac{1}{z^2}\right)dz}{z^2-2+\frac{1}{z^2}+9}=\int\frac{\left(1+\frac{1}{z^2}\right)dz}{\left(z-\frac{1}{z}\right)^2+9}$$

Again we put $t=z-\frac{1}{z}$ $\therefore$ $dt=\left(1+\frac{1}{z^2}\right)dz$

After substitution it becomes

$$\int\frac{dt}{t^2+3^2}=\tfrac{1}{3}\tan^{-1}\frac{t}{3}$$

$$=\frac{1}{3}\tan^{-1}\left(\frac{z-\frac{1}{z}}{3}\right)=\frac{1}{3}\tan^{-1}\left(\frac{z^2-1}{3z}\right)$$

$$=\tfrac{1}{3}\tan^{-1}\left(\frac{x}{3\sqrt{x+1}}\right).$$

(C) If L is linear and M is quadratic; we put $L=\frac{1}{z}$

The following example will illustrate the method.

Ex. *Integrate* $\int\frac{dx}{(2+x)\sqrt{4-x^2}}$

Let $2+x=\frac{1}{z}=z^{-1}$ $\therefore$ $dx=-\frac{1}{z^2}\,dz$ or, $x=\frac{1}{z}-2$

$\therefore$ $x^2=\left(\frac{1}{z}-2\right)^2=\frac{1}{z^2}-\frac{4}{z}+4$

or, $4-x^2=4-\frac{1}{z^2}+\frac{4}{z}-4=\frac{4}{z}-\frac{1}{z^2}.$

$$\therefore\quad I=\int\frac{-\frac{1}{z^2}\,dz}{\frac{1}{z}\sqrt{\frac{4}{z}-\frac{1}{z^2}}}=-\int\frac{\frac{1}{z^2}\,dz}{\frac{1}{z\,.\,z}\sqrt{4z-1}}$$

$$=-\int\frac{dz}{\sqrt{4z-1}}=-\int(4z-1)^{-\frac{1}{2}}\,dz$$

$$=-\tfrac{1}{4}\frac{(4z-1)^{\frac{1}{2}}}{\frac{1}{2}}=-\frac{2}{4}\sqrt{4z-1}$$

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$$= -\tfrac{1}{2}\sqrt{4\left(\frac{1}{2+x}\right)-1} = -\tfrac{1}{2}\sqrt{\frac{4-2-x}{2+x}}$$

$$= -\tfrac{1}{2}\sqrt{\frac{2-x}{2+x}}.$$

Otherwise :—

$$I=\int\frac{dx}{(2+x)\sqrt{4-x^2}}$$

Let $x=2\cos\theta$ Hence, $dx=-2\sin\theta\, d\theta$

Thus. $$I=\int\frac{-2\sin\theta d\theta}{(2+2\cos\theta)\sqrt{4-4\cos^2\theta}}$$

$$=-\frac{2}{2}\int\frac{\sin\theta d\theta}{(1+\cos\theta)\sqrt{4\sin^2\theta}}$$

$$=-\int\frac{\sin\theta d\theta}{(1+\cos\theta)\,2\sin\theta}=-\tfrac{1}{2}\int\frac{d\theta}{1+\cos\theta}$$

$$=-\tfrac{1}{2}\int\frac{d\theta}{2\cos^2\frac{\theta}{2}}=-\tfrac{1}{4}\int\sec^2\frac{\theta}{2}d\theta=-\tfrac{1}{4}\frac{\tan\frac{\theta}{2}}{\frac{1}{2}}$$

$$=-\tfrac{1}{2}\sqrt{\tan^2\frac{\theta}{2}}=-\tfrac{1}{2}\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}$$

$$=-\tfrac{1}{2}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}=-\tfrac{1}{2}\sqrt{\frac{1-\frac{x}{2}}{1+\frac{x}{2}}}$$

[$\because$ $x=2\cos\theta$]

$$=-\tfrac{1}{2}\sqrt{\frac{2-x}{2+x}}.$$

(D) If L and M are both quadratic, we put $\sqrt{\frac{M}{L}}=z$.

The following example will illustrate the method,

Ex. *Integrate* $\int\frac{dx}{(1+x^2)\sqrt{1-x^2}}$.

[... M. U. 72H, Bh. U. 65H]

Let $I=\int\frac{dx}{(1+x^2)\sqrt{1-x^2}}$

Let $\sqrt{\frac{1-x^2}{1+x^2}}=z$ or, $\frac{1-x^2}{1+x^2}=z^2$

or, $(1-x^2)=z^2(1+x^2)=z^2+z^2x^2$

or, $1-z^2=x^2(1+z^2)$

or, $x^2=\frac{1-z^2}{1+z^2}$

$\therefore$ $2xdx=\frac{-2z(1+z^2)-(1-z^2)2z}{(1+z^2)^2}dz$

$=\frac{-2z-2z^3-2z+2z^3}{(1+z^2)^2}dz$ or, $2xdx=\frac{-4z}{(1+z^2)^2}$

$\therefore$ $dx=\frac{-2zdz}{(1+z^2)^2}\sqrt{\frac{1+z^2}{1-z^2}}$

Again, $1+x^2=1+\frac{1-z^2}{1+z^2}=\frac{2}{1+z^2}$

and, $1-x^2=1-\frac{1-z^2}{1+z^2}=\frac{2z^2}{1+z^2}$

$\therefore$ $I=-2\int\frac{zdz}{(1+z^2)^2}\cdot\frac{\sqrt{1+z^2}}{\sqrt{1-z^2}}\cdot\frac{1+z^2}{2}\cdot\frac{\sqrt{1+z^2}}{\sqrt{2}\,.\,z}$

$=-\frac{1}{\sqrt{2}}\int\frac{dz}{\sqrt{1-z^2}}=-\frac{1}{\sqrt{2}}\sin^{-1}z$

$=-\frac{1}{\sqrt{2}}\sin^{-1}\sqrt{\frac{1-x^2}{1+x^2}}$